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LETTER TO THE EDITOR

$1/\sigma$ expansion for critical concentrations in the diluted spin glass

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Abstract. We study the threshold concentrations, p_q, for critical behaviour at zero temperature in the correlation functions $\chi^{(2q)}$ for an Ising spin glass in which nearest-neighbour interactions randomly assume the values $+J$, 0, and $-J$ with respective probabilities $p/2$, $1-p$ and $p/2$. Here $\chi^{(q)} \equiv [\Sigma_j \langle s_i s_j \rangle^q]_{\text{av}}$, where $\langle\ \rangle$ denotes an average over the ground states for a fixed configuration of J and $[\ ]_{\text{av}}$ an average over all such configurations. Due to frustration effects $p_c < p_2 < p_4 < p_6 \ldots < p_\infty$, where p_c is the percolation threshold. Thus spin-glass theory with only $\chi^{(2)}$ critical (at $p = p_2$) applies and the critical exponents along the $T = 0$ axis are the same as thermal critical exponents for $p = 1$. When the values $+J$ and $-J$ are replaced by distributions of narrow width, the frustration is removed and percolation exponents are expected at $p = p_c$.

The use of the replica method [1, 2] to treat the critical properties of quenched random systems has had a long history of varying success. When applied to ferromagnetic systems with weak disorder where randomness does not affect the qualitative nature of the thermodynamic phase, the results [3–5] have been seemingly quite satisfactory, in that they agreed with similar calculations not involving replicas [6–8]. In contrast, the use of replicas [9] for spin-glass (SG) systems (for a review see [10]), where randomness crucially affects the *nature* of the ordered phase, has led to a vast technology in which esoteric schemes [11] are employed to circumvent the instabilities [12] which plague the simpler calculations in the ordered phase. The calculations of critical exponents via the ε-expansion renormalisation group [13, 14] are presumed to be correct, since they are based on calculations in the *disordered* phase, where replica-symmetry breaking need not be considered. With these facts in mind, one might imagine that a calculation of the critical behaviour of the *diluted* SG [15] in its disordered phase should be possible. The Hamiltonian for the model we consider is

$$H = -\tfrac{1}{2} \sum_{ij} J_{ij} s_i s_j \tag{1}$$

where s_i assumes the values $+1$ and -1 and each $J_{ij} = J_{ji}$ is a random variable, non-zero only for nearest-neighbouring sites i and j, which assumes the values $-J$, 0, and $+J$ with the respective probabilities $p/2$, $1-p$, and $p/2$. Qualitatively, we would expect the phase diagram for this diluted SG to be as shown in figure 1. This diagram has an obvious similarity to that for the randomly diluted Ising model studied some time ago by Stephen and Grest [16], who initiated a method of calculation from which the

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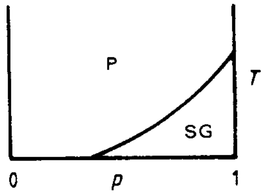

Figure 1. Phase diagram for the diluted SG as a function of temperature T and bond concentration p. Here SG denotes a phase with long-range SG order and P is the disordered phase.

critical behaviour along and near the $T=0$ axis could be obtained. They found that the critical exponents along the $T=0$ axis were those of the percolation problem [17, 18], since at $T=0$ the Ising model correlation functions $\chi^{(q)} = \Sigma_j\, [\langle s_i s_j \rangle^q]_{\mathrm{av}}$, where $\langle\,\rangle$ denotes an average over the ground states for a fixed configuration of J and $[\;]_{\mathrm{av}}$ denotes an average over all such configurations, reduce to the pair connectedness correlation function of the percolation problem.

Subsequently, Aharony [19] and Giri and Stephen [20] used the Stephen and Grest [16] formalism to study the critical behaviour of the diluted SG model along the $T=0$ axis. Both groups concluded that the critical properties of the diluted SG when the ordered phase was approached along the $T=0$ axis were described by the $\frac{1}{2}$-state Potts model. Basically, this result came about because, instead of all $\chi^{(q)}$ being simultaneously critical (as is the case for the dilute Ising model [16]), only $\frac{1}{2}$ of these, i.e. the $\chi^{(q)}$ with q even, were considered to be simultaneously critical at the percolation threshold, $p=p_c$. It was soon realised, however [20-22], that these calculations did not properly treat the subtle frustration effects [23] within this model typified by the system of figure 2. Half the configurations of this system are unfrustrated (i.e. the product of J around the square is positive) and cause no problems for the replica method. However, for the other half of the configurations the product of the J is negative, the ground state becomes non-trivially degenerate, as shown in figure 2, and the ground-state correlations assume the values given in the caption to figure 2. In particular note that spins across a diagonal are completely uncorrelated when the square is frustrated. Similar comments apply for more complicated systems. These ground-state effects are hard to reproduce within the replica formalism [19, 20], although for a small system one can take the replica limits so as to obtain correct quenched averages. Giri and Stephen [20] stated that their replica calculation led to a SG state with 'only a small amount of frustration'. Aharony and Pfeuty [22] recognised this equivalence to the $\frac{1}{2}$-state Potts model to be incorrect, due to the incorrect order of taking the $n\to 0$ and $T\to 0$ limits [21]. They concluded that in the absence of frustration this identification of the dilute SG with the $\frac{1}{2}$-state Potts model should hold.

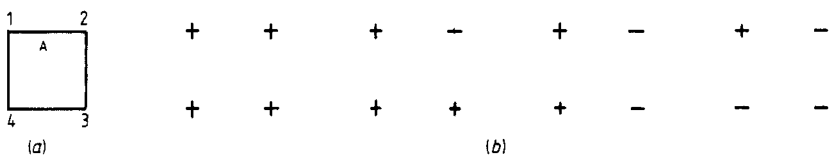

Figure 2. (*a*) A configuration with one antiferromagnetic bond (A) causing frustration in a single plaquette. All interactions (bonds) have the same magnitude. (*b*) The four ground states of the system for spin number 1 up, from which one gets $\langle\sigma_1\sigma_2\rangle = -\frac{1}{2}$, $\langle\sigma_1\sigma_3\rangle = 0$, $\langle\sigma_1\sigma_4\rangle = \frac{1}{2}$, etc.

Nevertheless, the actual nature of the critical behaviour of the model (1) with its attendant frustration remainded unclear.

The object of the present letter is to show that (i) the critical properties of the model (1) as the ordered phase is approached by increasing p along the $T=0$ axis are the same as those for the thermal SG transition at $p=1$ for which ε-expansion results exist [13, 14], and that (ii) an unfrustrated version of (1) at $T=0$ is in the same universality class as ordinary percolation.

To start, we argue that all the $\chi^{(2q)}$ have different critical threshold concentrations, which are greater than p_c. Since the *density* of squares is finite at $p=p_c$, it is intuitively clear that frustration prevents SG order from propagating over the incipient infinite cluster at $p=p_c$. To quantify this observation, we use a $1/\sigma$ expansion where $\sigma=z-1$, where z is the coordination number of the pure lattice, to calculate the threshold concentration p_{2q} where $\chi^{(2q)}$ diverges. We consider the low concentration expansion for $\chi^{(2q)}$ which we write in the form

$$\chi^{(2q)}(p)=\sum_{\Gamma} W(d,\Gamma)p^{n(\Gamma)}\chi_c^{(2q)}(\Gamma) \tag{2}$$

where the sum is over all topologically inequivalent connected clusters Γ, where $W(d,\Gamma)$ is the number of ways per site a cluster topologically equivalent to Γ can be formed on a hypercubic lattice in d spatial dimensions, $n(\Gamma)$ is the number of bonds in the cluster Γ, and $\chi_c^{(2q)}(\Gamma)$ is the cumulant value of the susceptibility for the cluster Γ which is obtained recursively from the bare susceptibility by

$$\chi_c^{(2q)}(\Gamma)=\chi^{(2q)}(\Gamma)-\sum_{\gamma\subset\Gamma}\chi_c^{(2q)}(\gamma) \tag{3}$$

where $\chi^{(q)}(\Gamma)=[\Sigma_{ij\in\Gamma}\langle s_is_j\rangle^q]_{\mathrm{av}}$. Aharony and Binder [23] estimated p_2 from the series (2) for $\chi^{(2)}$, but they did not have enough terms to determine whether or not p_2 differed from p_c. We analyse (2) for general q by considering the large-σ limit. A simplifying feature at large σ is that $W(d,\Gamma)/\sigma^{n(\Gamma)}$ contains a relative factor $(1/\sigma)^k$ for each loop of $2k$ bonds [24, 25]. Similarly, corrections in $W(d,\Gamma)$ from values on the Cayley tree (which has no loops) can be classified in powers of $1/\sigma$ depending on the loops involved implicitly in the corrections. Another simplification is that the cumulant susceptibility in (2) vanishes for any diagram with more than two free ends. Thus to order $(1/\sigma)^3$ we only need to consider contributions from the diagrams shown in figure 3, whose weights $W(d,\Gamma)$ to this order are given in table 1. A calculation of the susceptibilities $\chi^{(2q)}(\Gamma)$ follows closely from the discussion of Aharony and Binder [23] as illustrated in figure 2 and the results are given in table 1 for the relevant diagrams shown in figure 3. The $\chi^{(2q+1)}(p)$ have trivial properties ($\chi^{(2q+1)}(p)=1$) due to the fact that J is governed by a probability distribution which is even in J.

We now collect the results into the form $\chi^{(2q)}(p)=\Sigma_n a_n(2q,\sigma)p^n$, with

$$\sigma^{-n}a_n(2q,\sigma)=[(\sigma+1)/\sigma](1-n/\sigma^2-2n/\sigma^3)+(1-3/\sigma)[-\tfrac{9}{4}+\tfrac{1}{2}(\tfrac{1}{2})^{2q}]n/\sigma^2$$
$$+[-15+2(\tfrac{2}{3})^{2q}+2(\tfrac{1}{3})^{2q}]n/\sigma^3+\mathrm{O}(1/\sigma^4) \tag{4}$$

where we have dropped terms independent of n in the corrections to the leading term, since they will not affect our results. It appears that all the a_n are positive, at least for large σ.

Accordingly, we may determine p_{2q} by $p_{2q}=\lim_{n\to\infty}[1/a_n(2q,\sigma)]^{1/n}$, so that

$$p_{2q}=(1/\sigma)\{1+(1/\sigma^2)[\tfrac{13}{4}-\tfrac{1}{2}(\tfrac{1}{2})^{2q}]+(1/\sigma^3)[\tfrac{41}{4}+\tfrac{3}{2}(\tfrac{1}{2})^{2q}-2(\tfrac{2}{3})^{2q}-2(\tfrac{1}{3})^{2q}]\} \tag{5}$$

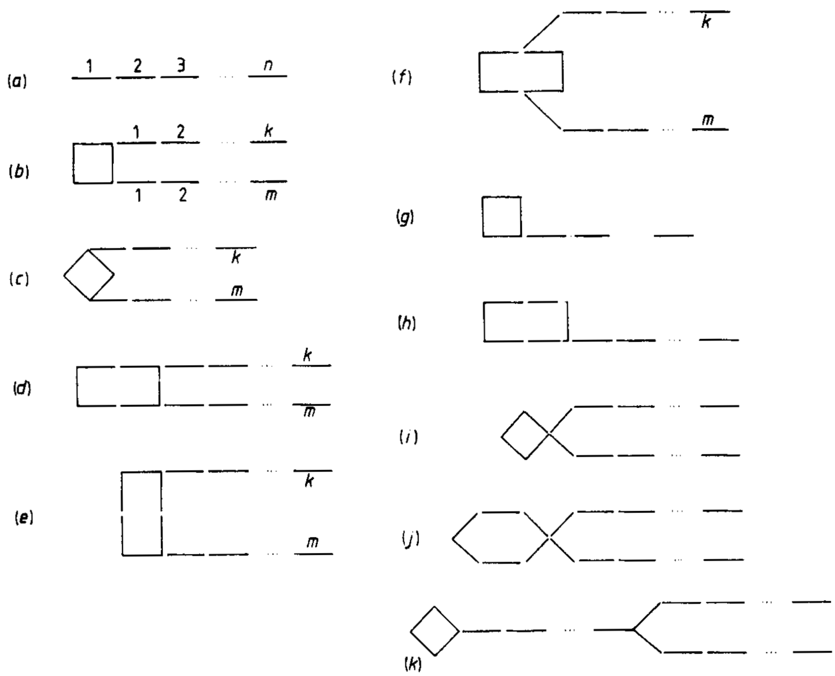

Figure 3. Clusters which potentially contribute to the susceptibilities up to order $1/\sigma^3$. Clusters (a)-(f) have contributions given in table 1 which contribute to (4). Clusters (g) and (h) give contributions to (4) which are independent of n and therefore are dropped. Clusters (i)-(k) have zero cumulant susceptibilities and are not listed in table 1.

Table 1. Weights and cumulant susceptibilities for the clusters of figure 3.

Diagram, Γ	Weight[a] $W(\Gamma)$	$\chi_c^{(2q)}(\Gamma)$
(a)	$\dfrac{\sigma+1}{2\sigma}\sigma^n\left(1-\dfrac{n}{\sigma^2}-\dfrac{2n}{\sigma^3}\right)^{\text{b}}$	2
(b)[c]	$\dfrac{\sigma^2-1}{2}\left(\dfrac{\sigma-1}{\sigma}\right)^2\sigma^{k+m}$	$-3+(\frac{1}{2})^{2q}$
(c)[c]	$\dfrac{\sigma^2-1}{4}\left(\dfrac{\sigma-1}{\sigma}\right)^2\sigma^{k+m}$	-3
(d)[c]	$2\sigma^{k+m+3}$	$-3+(\frac{2}{3})^{2q}$
(e)[c]	$2\sigma^{k+m+3}$	$-3+(\frac{1}{3})^{2q}$
(f)[c]	σ^{k+m+3}	-3
(g)	$\dfrac{\sigma^2-1}{2}\left(\dfrac{\sigma-1}{\sigma}\right)\sigma^n$	Not needed
(h)	$2\sigma^{n+3}$	Not needed

[a] Correct to order $1/\sigma^3$ for large n or $m+k$.
[b] See [22]. Correction terms independent of n are omitted.
[c] To count all contributions, set $k+m=n$ and sum k from 1 to $n-1$. (This takes account that the symmetry factor of the diagram for $k=m$ differs from that for $k\neq m$.)

which should be compared to the result [26] for the percolation threshold p_c:

$$p_c = (1/\sigma)(1 + 5/2\sigma^2 + 15/2\sigma^3). \tag{6}$$

Our result (5) shows that

$$p_c < p_2 < p_4 < p_6 < \ldots < p_\infty . \tag{7}$$

For $d = 6$ ($\sigma = 11$), (6) and (5) give $p_c = 0.0933$, $p_2 = 0.0939$ and $p_\infty = 0.0942$. In this calculation one can see quite clearly the role of frustrated configurations in reducing the magnitude of a_{2q}, which causes p_{2q} to increase with increasing q. For the dilute Ising model $\lim_{q\to 0} \chi^{(q)}$ becomes equal to the percolation correlation function even for $T \neq 0$ [27]. Here $p_c < p_{q\to 0}$ because for frustrated plaquettes $\langle s_i s_j \rangle$ can vanish even when sites i and j are connected, as illustrated in figure 2. This can also be seen from table 1, where the $\chi_c^{(2q)}$ for $q \to 0$ reproduce percolation, except for diagrams (c) and (f) where such correlations actually vanish for frustrated configurations.

What conclusions can be drawn from the result (5)? As p is increased through the value $p = p_2$, the $\chi^{(2q)}$ with $q > 1$ are not critical because $p_{2q} > p_2$. Thus, in contrast to the dilute Ising model [16] where all the $\chi^{(q)}$ are simultaneously critical for $p = p_c$, here only $\chi^{(2)}$ becomes critical as p approaches p_2. Therefore the critical properties of this model are the same as in the usual thermal transition for the SG at $p = 1$ for which only $\chi^{(2)}$ is critical [13]. In a sense frustration mimics temperature in that it leads to an unstable perturbation which drives the system away from the fixed point where all the $\chi^{(2q)}$ are equal. It must be admitted that the values of the exponents determined by Aharony and Binder seem to be definitely smaller than those for the SG at $p = 1$ as determined by high-temperature series [28]. However, the series for $T = 0$ is rather short (having eight terms) and it could easily be influenced by the crossover from the $\frac{1}{2}$-state Potts model fixed point which is nearby. (Note that, although the p_{2q} do depend on q, this dependence is rather weak.)

Furthermore, one can ask under what conditions it might be possible to attain simultaneous criticality of all the $\chi^{(2q)}$. This will happen if there is no frustration [22], which is the case if we take each bond to have a quenched random exchange constant J governed by the distribution

$$P(J) = (1-p)\delta(J) + pf(J)/2 + pf(-J)/2 \tag{8}$$

where $f(J) = 1/(2\Delta)$ for $|J-1| < \Delta$ and $f(J) = 0$ otherwise, with $\Delta \ll 1$. In this case, frustrated configurations occur with essentially zero probability [29], since they require a precise balance of J in different branches of a loop. Thus at $T = 0$ one has $|\langle s_i s_j \rangle| = 1$ for essentially all configurations. In this case, all the $\chi^{(2q)}$ receive unit contributions from sites which are in the same cluster and zero contributions otherwise. As a result, $\chi^{(2q)}$ is equivalent to the percolation pair-connectedness correlation function. It is then clear that the critical properties of the model (8) for $T = 0$ are the same as those of the percolation problem [17, 18]. Thus we see no simple scenario in which one can identify the bond-diluted SG with the $\frac{1}{2}$-state Potts model. However, our discussion does indicate another interesting question, namely it would be of interest to discuss the crossover which must occur in the model (8) as the temperature is raised from 0 to the order of $\Delta \ll 1$. When the temperature is of order Δ, then averages over a single ground state will effectively be replaced by averages over the frustrated plaquettes. Note that this crossover occurs in the variable (T/Δ) and not in an exponential variable $\exp(-J/T)$, as occurs in the diluted Ising model [16]. In fact, it is this crossover which ultimately may have to be understood in order to deal properly with the SG phase.

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References

[1] Edwards S F and Anderson P W 1975 *J. Phys. F: Met. Phys.* **5** 965
[2] Emery V J 1975 *Phys. Rev.* B **11** 239
[3] Grinstein G 1974 *AIP Conf. Proc.* **24** 313
[4] Grinstein G and Luther A 1976 *Phys. Rev.* B **13** 1329
[5] Khml'nitsky D E 1975 *Zh. Eksp. Teor. Fiz.* **68** 1960
[6] Harris A B and Lubensky T C 1974 *Phys. Rev. Lett.* **33** 1540
[7] Harris A B and Lubensky T C 1974 *AIP Conf. Proc.* **24** 311
[8] Lubensky T C 1975 *Phys. Rev.* B **11** 3573
[9] Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **35** 1792
[10] Binder K and Young A P 1986 *Rev. Mod. Phys.* **58** 801
[11] Parisi G 1979 *Phys. Rev. Lett.* **43** 1754
[12] Bray A J and Moore M A 1979 *J. Phys. C: Solid State Phys.* **12** 79
[13] Harris A B, Lubensky T C and Chen J-H 1976 *Phys. Rev. Lett.* **36** 415
[14] Green J E 1985 *J. Phys. A: Math. Gen.* **18** L43
[15] Palmer R G and Bantilan F T 1985 *J. Phys. C: Solid State Phys.* **18** 171
[16] Stephen M J and Grest G S 1977 *Phys. Rev. Lett.* **38** 567
[17] Harris A B, Lubensky T C, Holcomb W K and Dasgupta C 1975 *Phys. Rev. Lett.* **35** 327, 1397(E)
[18] de Alcantara Bonfim O F, Kirkham J E and McKane A J 1980 *J. Phys. A: Math. Gen.* **13** L247; 1981 *J. Phys. A: Math. Gen.* **14** 2391
[19] Aharony A 1978 *J. Phys. C: Solid State Phys.* **11** L457
[20] Giri M R and Stephen M J 1978 *J. Phys. C: Solid State Phys.* **11** L541
[21] De Dominicis C and Stephen M J 1978 *J. Phys. C: Solid State Phys.* **11** L969
[22] Aharony A and Pfeuty P 1979 *J. Phys. C: Solid State Phys.* **12** L125
[23] Aharony A and Binder K 1980 *J. Phys. C: Solid State Phys.* **13** 4091
[24] Fisher M E and Gaunt D S 1964 *Phys. Rev.* **133** A224
[25] Harris A B and Meir Y 1987 *Phys. Rev.* A **36** 1840
[26] Gaunt D S and Ruskin H 1978 *J. Phys. A: Math. Gen.* **11** 1369
[27] Harris A B 1983 *Phys. Rev.* B **28** 2614
[28] Singh R R P and Chakravarty S 1986 *Phys. Rev. Lett.* **57** 245
[29] Sherrington D and Schwartz M 1985 *J. Phys. C: Solid State Phys.* **18** L817